Pre-publication copy: Carruthers, E. and Worthington, M. (2008) Children's mathematical graphics: young children calculating for meaning. In I. Thompson. (Ed.) *Teaching and learning early number*. (Second Edition. Maidenhead: Open University Press.

Children's mathematical graphics: young children calculating for meaning

Elizabeth Carruthers and Maulfry Worthington

Introduction

To understand and use the standard written language of mathematics effectively children need to build on their earliest understanding of relationships between objects, signs and meanings. This relationship originates in research on semiotics (the symbol-meaning relationship) and was explored by the Russian psychologist Vygotsky in his research on children's early writing (1978).

However is clear that children's early written mathematics often cause difficulties, resulting in only superficial understanding. Research has highlighted the fact that children as old as nine years of age who used standard mathematical symbols every day in school, were reluctant to represent addition and subtraction with the same symbols, when given a situation other than a page of 'sums' (Hughes, 1986).

Some of these difficulties may relate to a lack of clarity about the pedagogy: as one nursery teacher explained to us, 'between the ages of three and seven there is a no man's land in the teaching of written mathematics'. This dilemma is confirmed by our findings from two studies we made with over five hundred teachers (Carruthers and Worthington, 2006).

The theoretical basis underpinning this chapter is socio-cultural: children's homes, communities and early childhood settings provide authentic contexts in which meanings are made and negotiated.

Underpinning context

Using clinical studies in which young children played an invented game with tins, Hughes's significant research (1986) broke new ground. His findings revealed how young children could use their own marks and symbols to represent and communicate quantities in personally meaningful ways. Hughes's findings suggested that this might reduce some of the difficulties that young children experience with early written mathematics, although the children's development from their earliest marks through to standard written mathematics was not a feature of his research.

Publication of Hughes's book coincided with interest in 'emergent' writing approaches in England and by 1990 there were the beginnings of interest in 'emergent mathematics'. In both emergent approaches children build on their earliest understandings of written symbols in meaningful contexts with sensitive adult support. However rather than exploring letter-symbol and sound correlation to communicate meanings, children's mathematical marks and representations carry meanings about quantities and the number system.

In a study with young children Munn (1997) repeated Hughes's 'tins game' and explored the relationship between emergent writing and emergent mathematics. Identifying *communicative purposes* as an important feature of emergent writing, she questioned the extent to which the children understood that they could use their symbols to communicate meanings about quantities. She found the exception were children who used standard written numerals they had been taught, arguing that such learning:

> is set in the context of interaction that closely conveys the quantitative these symbols. When children invent ways of depicting quantity based on drawing, this is not necessarily so… (Pre-schoolers)… were unlikely to have had any conversations about the meaning of their activity simply because such activity has no widespread cultural meaning', (Munn, 1997, p.94).

The children's difficulty appeared to stem from the fact that they were unlikely to have engaged in *dialogue* with adults about their symbols to negotiate 'cultural meanings'. And unlike literacy in the nursery and school, mathematics is not always embedded in meaningful social contexts that can help children make sense of the value of symbols as communicative tools.

Against these related backgrounds of emergent writing and Hughes's research, 'emergent mathematics' appeared to offer a positive way of supporting young mathematicians.

Children's mathematical graphics

As teachers we were experienced with the pedagogy of emergent writing, valuing its benefits for children's learning: however we eventually rejected the term *emergent mathematics* and not only because these two symbolic languages are inherently different.

In the 1990s the term *emergent writing* had developed negative connotations when sometimes Reception or Year 1 teachers lacked professional pedagogical knowledge to support children in progressing towards standard forms. We also wanted to use a term that went beyond suggesting that children's understanding *emerges* on its own with little pedagogical support.

It was clear that the numerous examples of children's mathematical marks and representations we had collected (scribble- marks; drawings; personal and standard symbols; letters, words and numerals) could be described as 'graphics', leading us to originate the term *children's mathematical graphics.*

Developing Meanings

Central to our thinking about children's mathematical graphics is the premise that from birth children are constantly trying to make meaning of their world: rooted in a social cultural perspective children use whatever they can within the culture of their own lives to make personal meaning. This view of cognition moves:

> beyond the idea that development consists of acquiring skills. Rather, a person develops through participation in an activity, changing to be involved in the situation at hand in ways that contribute both to the ongoing event and to the person's preparation for involvement in other similar events' (Rogoff, 2003, p.254).

The belief is that all higher order functions such as learning grow out of social interactions.

Symbolic tools

One of the most significant aspects of children's development that Vygotsky explored was his research on 'symbolic tools'; symbol systems that include language; writing; diagrams; maps; algebra and counting and road signs. Just as hammers, screwdrivers and spades aid physical tasks; *symbolic tools* mediate learning and help us resolve cognitive problems. Indeed, mathematics as a subject has been described as 'really a matter of problem solving with symbolic tools' (van Oers, 2001, p. 63).

In school the standard symbolic tools of mathematics include numerals and operators, leading to standard calculations. However in addition to Hughes, a number of other researchers have identified written mathematics as the main aspect of mathematics that young children find difficult. Ginsberg observed that 'while they make many errors in *(standard)* written arithmetic, children may in fact possess relatively powerful informal knowledge. This can be used as the basis for effective instruction,' (Ginsberg, 1977, p.129). Yet children's learning and the pedagogy of early written mathematics remain poorly understood.

Our research has focused on effective ways of building on children's 'powerful informal knowledge'. Its success allows bridging of the 'gap' in understanding between informal and standard written mathematical symbols, identified by Hughes (1986). Effective learning cultures in which children's representations are valued, and where collaborative dialogue about their meanings and symbols feature highly.

Children's Mathematical Thinking

Vygotsky observed that there is 'a unified historical line that leads to the highest forms of written language (1978, p.116). He traced young children's understanding from their symbolic (pretend) play, identifying their appropriation of objects to which they gave alternative meanings. We follow this line of development through to calculations.

Children's own mathematical graphics are 'thinking on paper' and uncovering their meaning is crucial in supporting their journeys towards conventional mathematical symbols. The marks and representations that children use reveal their thinking. Their graphics are personal thinking tools and key to their development: this is not *recording* following a practical activity; for example drawing the cubes following their use in a practical context. Children's own mathematical marks and representations also assist them in moving from mathematics with concrete materials into more abstract mathematical thought and symbols.

We found that when children are given the opportunity to use paper and pens to support their thinking, they develop effective personal strategies. Not only do these strategies work for them in solving mathematical problems but much more importantly they understand what they do (Carruthers and Worthington, 2006). We want children to understand written mathematics rather than hurdling blindly through a set of 'tricks' that they have memorised.

Uncovering children's development

Perhaps the most illuminating aspect of our research on *children's mathematical graphics* is that the data revealed a pattern of development. As we sorted through the 700 examples of children's own written mathematics, significant categories were revealed, leading to the taxonomy tracing development from birth to eight years.

See separate sheet for taxonomy.

Figure 10.1: The development of written number and quantities

Explanations of each aspect of the taxonomy are outlined below.

Beginnings in Play

Making meanings

As young children explore their world they play in many ways with different objects, materials, and people. The work of Kress (1997) in describing and explaining his observations of children's *multi-modal* play provides the foundations of children's journey into all symbol systems including mathematics. In play children have available many modes of representation for make-believe: these include role play and small world play; block play and junk modelling. Kress's focus on making meanings 'with lots of different stuff' relates to Vygotsky's research on symbolic tools.

3-year-old Sol gave his teacher what appeared to be a piece of screwed up paper, declaring that it was a telescope. His teacher said 'Oh! Tell me about this' and pointing to a narrow crack in the paper Sol said 'Look! There's the hole'. Through this simple act Sol signified the act of 'looking through' a telescope.

When children are engaged in free, child-initiated play with an abundance of open materials such as cloth, cardboard boxes, cubes, cylinders, buttons and straws; they create personal meanings: the materials and objects they choose then become all sorts of things with which they can play. Meanings often take on a fluid quality and children transform them from day to day (Worthington, in Moyles, 2007).

Children also make meanings with marks in their play. In the nursery 4-year-old Naadim was playing 'shops'. Seeing some paper and pens nearby, he made some marks across a page that resembled letter 't' and then read what he clearly intended to be a shopping list: 'spaghetti, cucumber and potatoes'. With his list in his hand Neil picked up a shopping basket and went shopping. Naadim's mother told his teacher that her son shopped with his parents regularly and that they wrote a shopping list to take with them: in his play Naadim had drawn on this important cultural aspect of experience.

The development of written number and quantities

Early Exploration with marks

Young children's first marks, sometimes referred to as scribbles, are a major development in a child's step towards multi-dimensional representations of her world. Very early marks can be ignored by adults and scribbles may be seen as random accidents. However in his study of young children's drawings and paintings, Matthews (1999) regards these 'scribbles' as 'products of systematic investigation rather than haphazard actions'. Malcholidi, 1998, suggests that if children give meaning to their scribbles then they may be moving forward in their development of representational images: this is supported by our research.

Between the ages of 3- and 4-years children begin to differentiate between their marks for drawing and writing, and those to which they attach mathematical meanings. For example 3-year-old Matt was at home and his aunt was sitting at a table near Matt and writing postcards. Matt wanted to be a part of this literate activity and covered numerous pieces of paper with his marks: some he referred to as drawings and for some he ‘read’ their written content. On two pages Matt referred to mathematical meanings for his marks ‘reading’ the marks on one page as ‘I spell 80354’. His parents thought that his reference to spelling may have related to family talk about how to spell his brother’s name and that it was possible that he had heard someone refer to a string of numbers for a telephone number.

Figure 10.2: *Matt’s marks*

Early Written Numerals

Children refer to their marks as numbers and begin to explore ways of writing numerals. Their perception of numerals and letters is of symbols that mean something, first differentiated in a general sense; ‘this is my writing, ‘this is a number’. Young children’s marks gradually develop into something more specific when they name certain marks as numerals. At this stage their marks may not be recognisable as numerals but may have number-like qualities. This development is similar to the beginning of children’s early writing.

Numerals as labels

Young children are immersed in print and see symbols and texts in their environment, in the home, on television and their community. Children often attend to these labels and are interested in how they are used: they can write in contexts which make sense to them (Ewers- Rogers and Cowan, 1996). Children look at the function of written numerals in a social sense and by the time they enter school, understand the different meanings of numbers of numbers. In our research we found that children moved from knowing about these symbols in the environment to writing them down for their own purposes. This is a significant shift since when they chose to write these numbers they have converted what they read into a standard symbolic language and have chosen to use them in meaningful contexts.

Representing quantities that are not counted

Young children’s marks are often lively and at first give the sense of something without needing to be too exact. When children *represent quantities that are not counted* their graphics almost have an impressionist air about them as they focus on a particular aspect at the time. For example Joe, aged 3-years represented a spider with many legs and said the spider had 8 legs. Joe did not represent the 8 legs exactly; neither did he count the legs he had drawn. He had the sense of a spider and the many legs featured prominently in his drawing. This was an exciting and imaginative drawing of a spider because Joe was unrestricted by influences of school. When children *represent quantities that they do not count,* it is their personal sense of quantity that they represent.

Representing quantities that are counted

Our analysis of young children’s marks also showed that they *represent quantities that they count*. For example 3-year-old Jenna drew vertical lines with coloured pens naming them ‘raindrops’, and counting them when she had finished. Our evidence shows that *quantities that are not counted* precede those that they count, although there is often an overlap of these two aspects. This aspect of development leads children directly to the beginnings of written calculations.

Hughes's analysis of the children's representations in the 'tins game' (1986) revealed aspects of their development that we categorised as *representing quantities that are counted.*

The development of early operations: children's own written methods and strategies

See separate sheet for taxonomy.

Figure 10.3: *Early operations: the development of children's own written methods*

Written Calculations

The dimensions of children's development above provide the crucial foundations of written calculations. Children need to continue to have plenty of opportunities to explore their mathematical ideas on paper as they explore calculations.

As children move on to explore calculations in diverse ways, their own representations support their mental methods and help them work out calculations. Counting features a large part in the beginnings of written calculation and children's preference for counting persists even though schools may choose to present calculations with other strategies. The variety of graphical responses that children choose also reflects their personal mental methods and intuitive methods developed from 'counting all'.

Counting Continuously

We use this term to describe this aspect of children's early representations of calculation for addition and subtraction. Several studies have shown that young children can carry out simple additions and subtractions (with objects and verbally) and that they can do this through counting strategies: the most common strategy is to 'count all' or to count the final number of items (Carpenter and Moser, 1984).

Since the 'counting all' strategy is not one that children are taught, Hughes suggests that we can infer that it is a self-taught strategy (Hughes, 1986, p.35). When children are given a worksheet with two sets of items to add, they count the first set and then continue to count the second set. This is misleading for teachers because although such a page will be termed 'addition' children will only be using it to count and do not recognise the separation and combining of the sets.

However when children choose to represent addition by a *continuously count* they have begun to understand the need to separate the two sets and are developing a sense of addition by combining them. This is an important distinction and also a good reason to give children opportunities to use their own methods and representations. In the reception class Alison (figure 10.4) wanted her friends to each bring a teddy to the café that her class had organised and her teacher asked her to work out what the total would be (i.e. seven children plus seven toys). Counting the children and then their toys she represented the total using numerals. Alison decided to check what she had written: finding that she had written numerals to seventeen, she put brackets round the numbers she did not need. The hand she drew may denote the action of adding although we cannot be sure.

Figure 10.4 *Alison*

The 'melting pot'

As children explore different ways to calculate we found a variety of different representations of operations which we refer to as a 'melting pot'. Carpenter and Moser (1984) studied the range of strategies children use to add and subtract and found that a variety of informal counting strategies persist through the primary years. Children's choices of ways to represent operations do not remain static since discussion and modelling alter individuals' perspectives, introducing them to further possibilities. Since children seem to find the standard algorithms and calculation difficult to understand (Hughes 1986, Ginsberg, 1989) it not surprising that at this stage we can see children explore many strategies. It is important that children are free to work out their own sense of calculations in ways they understand. It is this continual striving for personal understanding that we must encourage in mathematics from the beginning.

Separating sets

We found that children use a range of strategies to show that the two amounts that they are adding (or subtracting) are separate, for example by:

- Grouping the two sets of items to be added, perhaps by leaving a space between them or by representing them on opposite sides of the paper
- Separating the sets with words
- Putting a vertical line between them
- Putting an arrow or personal symbol between the sets
- Drawing rings (boxes) around each set

These strategies indicate that children are moving into new ways of working.

Exploring symbols

As children explore the role and use of symbols, they use a range of personal or invented symbols, or approximations of standard symbols and their development is strengthened by three additional and significant strategies.

Additional strategies

Implicit symbols

Children who have been used to representing calculations in their own ways may have an understanding of ‘=’ or ‘+’ but have not represented these standard symbols. The marks they make or the arrangement of their calculation, show that the symbol is *implied*: their understanding is revealed when often they ‘read’ their calculation as though to include absent features. This is where mental and written calculations combine to make meanings fluid: children are not encumbered by set rules to write the calculation in one ‘correct way’ but combine everything they know. The examples in this section are from children in Reception.

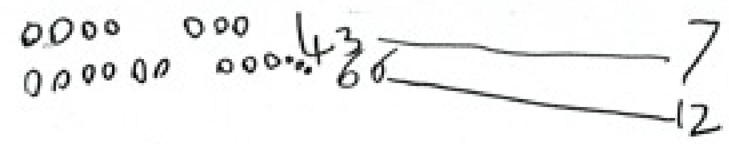

Figure 10.5 *Jack*

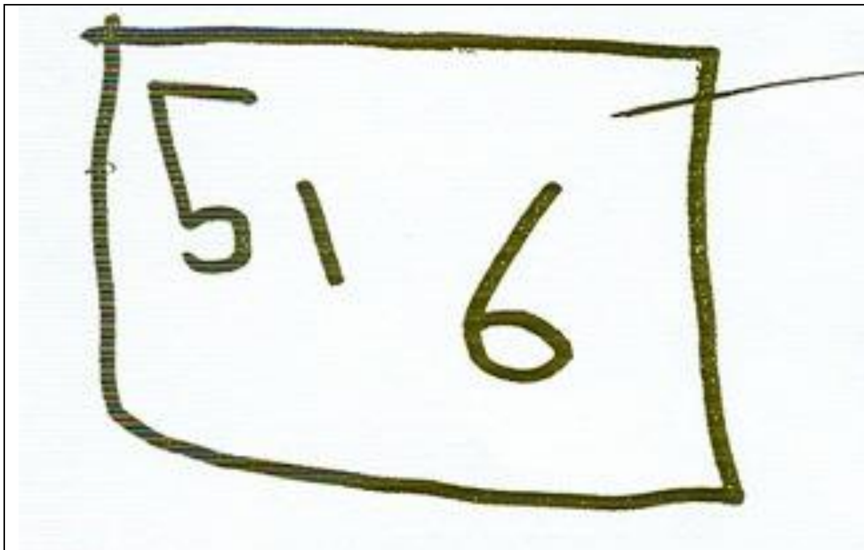

Figure 10.6 *William*

Figure 10.7 *Jax*

Some children omit the operator sign altogether: by leaving a space Jack (figure 10.5) implied the operant for addition (the 'doing' of the calculation) and used a single line for the equals sign. In figure 10.6 William also implied an equal's sign by leaving a space between the 1 and the 6 and by leaving a space between the six and the four dots she had drawn, Jax implied the subtraction sign; finally adding a letter 'T' to signify 'ten' (figure 10.7).

These examples are different but all show that the children were able to work out the calculation, and know the function of the missing symbol although they chose not include one. At this point in their development their calculation skills are in advance of their written knowledge of standard operators. This has parallels with very early counting and when children are sometimes pressurised into writing the correct numeral, they may be unable to write numerals in standard forms, resulting in adults failing to recognise their current number knowledge. The move in England to mental methods has helped resolve this to some extent, since assessment of children's early mathematical knowledge is judged more on their ability to calculate mentally. This is particularly pertinent for very young children when they first enter an educational setting, when their mathematical knowledge may not always be acknowledged and teachers may over-emphasize skills and correctly written numerals rather than uncover children's existing mathematical knowledge.

Code-switching

Code-switching is a term used in second-language learning: as learners develop fluency in their second language they substitute features of their first language (such as familiar words) for those they do not yet know. In their early written mathematics we found many examples when children *code-switch* within calculations, combining drawings, writing; personal standard symbols. This allows them to use ways of representing that support their thinking and move towards using increasingly standard forms of calculations. John (figure 10.8) was adding the grapes in two dishes that he had chosen (and that he would eat after he had worked out their total). He chose to use a written response to help his thinking, switching between writing and numerals: ‘2 grapes, there is two; 4 grapes there is four’ before adding the total (6) on the left of the page. Learning about effective layout develops through collaborative dialogue as children’s understanding grows.

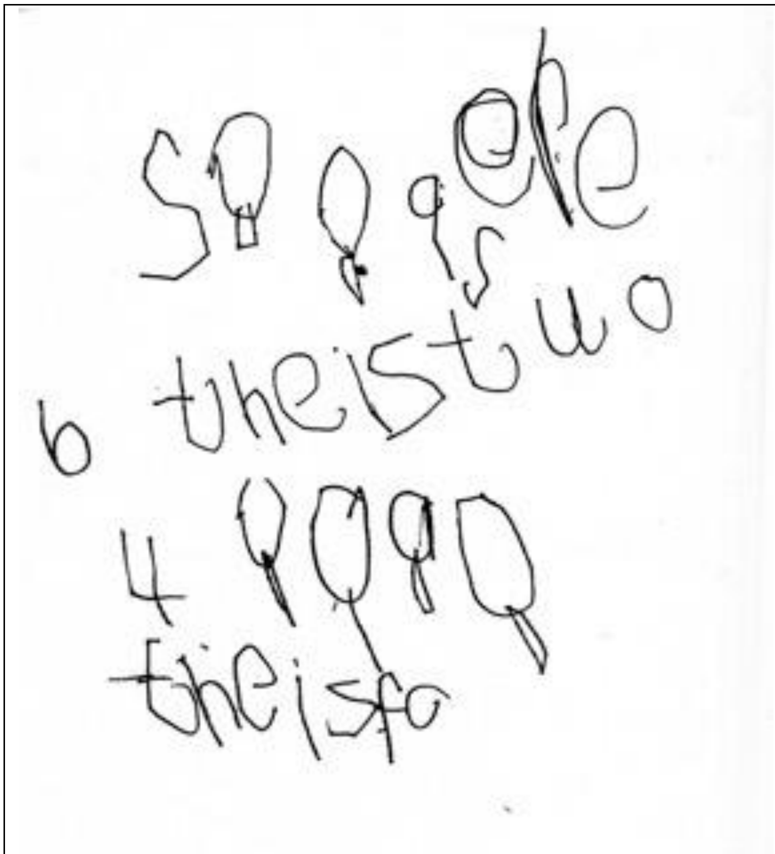

Figure 10.8 *John’s grapes*

Narrative action

Narrative action is a particularly significant feature: our term describes individuals' decision to include either hands or arrows in their calculation to signify the operant. This feature has been identified by other researchers and which they term *dynamic schematization*: 'such representations indicate 'higher levels of understanding' since they have the potential to show 'the relationship between drawing and object, sign and meaning transformation' (within calculations) (Poland and Van Oers, 2007p. 271/272).

Narrative action appears to support internalised *mental* images of earlier concrete operations of addition or calculation: the hand or arrows they include allow them to reflect on the operant and its function. In the following activity the children were in their first term in Y1. In figure 10.9, Barney drew an arc of arrows signifying his former action of 'taking away' to support his thinking about a subtraction game with beans.

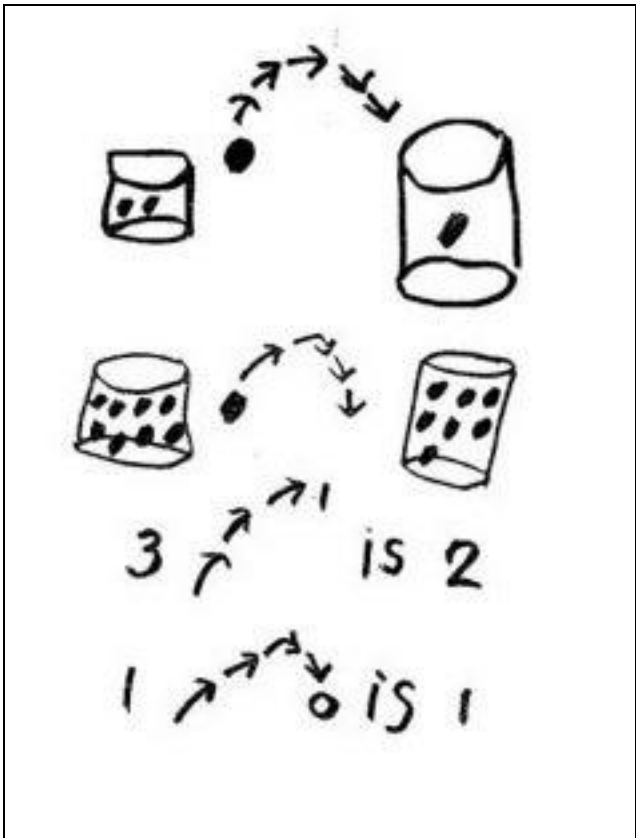

Figure 10.9 *Barney – subtracting beans*

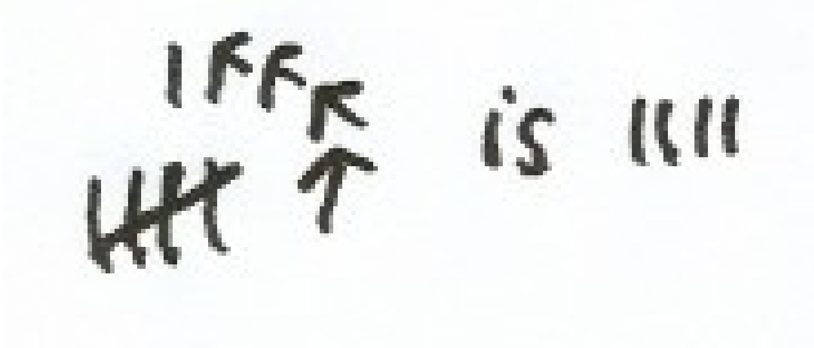

Figure 10.10 *Kristian*

Other children chose to explore *the* hands and arrow symbols that their peers had introduced. Playing the beans subtraction game, Kristian explored Barney's arrows-symbol in his own way (In figure 10.10).

Playing the same game, Alex (figure 10.11) introduced the idea of using a hand to signify the same 'taking away' action and Barney explored Alex's hand-symbol as an alternative to his arrows (figure 10.12).

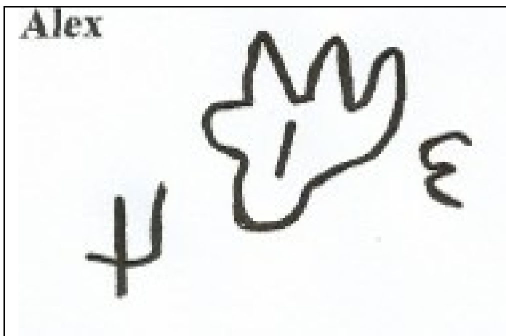

Figure 10.11 *Alex*

Figure 10.12 *Barney*

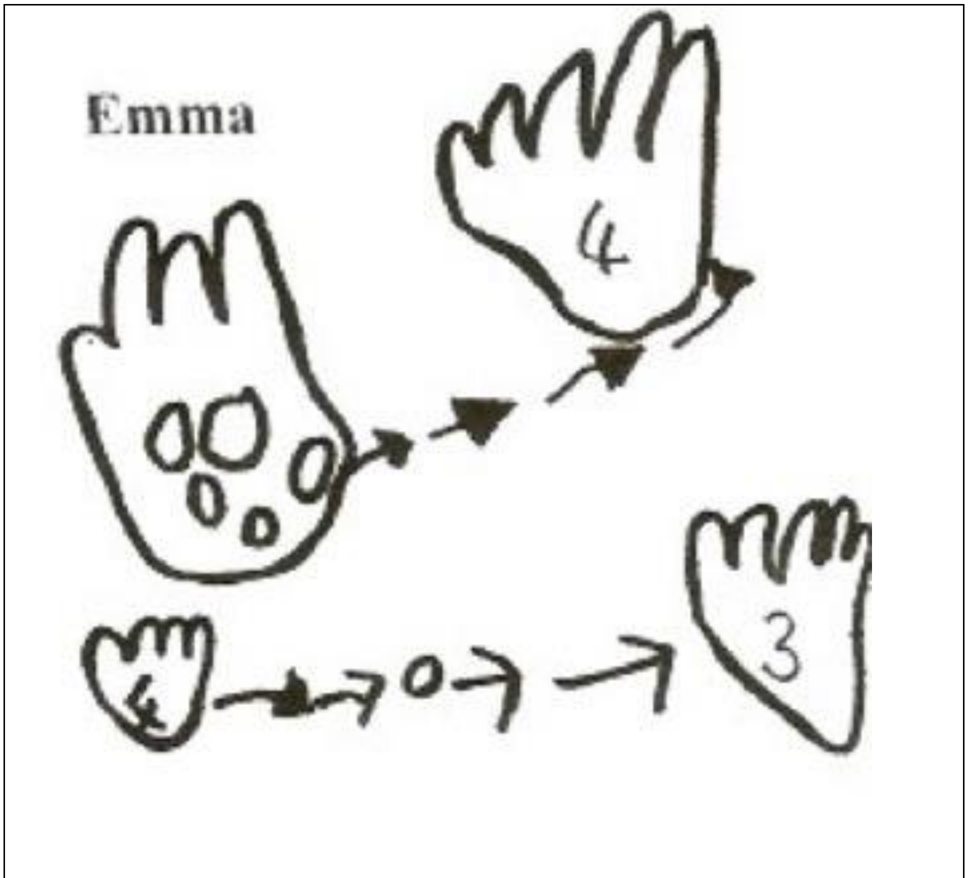

Figure 10.13 *Emma*

In an interesting adaptation, Emma used the hands symbol and made it her own by combining it with arrows (figure 10.13).

Through using narrative action to signify the operation, children are effectively moving into their *zone of proximal development* or ‘ZPD’, which Vygotsky describes as ‘*the distance between the actual development level as determined by independent problem solving, and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers’* (Vygotsky, 1978, p. 86; italics in the original).

Narrative actions appear to serve a dual purpose:

- They have an important *bridging function* between concrete and mental operations; between physical *actions* with concrete materials and the operators within written calculations
- They enable children to move from exploring a range of personal symbols for operators, to understanding what operators signify and their function within a calculation.

Standard symbolic operations with small numbers

By this point children have already been introduced to standard symbols and ways of working through a range of models (see Carruthers and Worthington, 2005, chapter 10). They understand the complex roles and functions of various mathematical symbols and have developed a wide range of strategies that they can use flexibly to solve problems.

In a mixed Reception / Y1 class the teacher had planned a dice game for children to play in pairs. Individuals playing the game rolled the two dice in turn, providing an opportunity for children to either count or add the dots and to use paper to help them think about the symbols if they wished.

Amelie had just started school and explored her ideas in a very spontaneous way. Her teacher observed and listened as she carefully counted the dots each time and then counted out loud as she marked them on the right of her paper. In her first term in Y1, Anna chose to use the opportunity to use standard symbols in horizontal written calculations.

Figure 10.14 *Amelie*

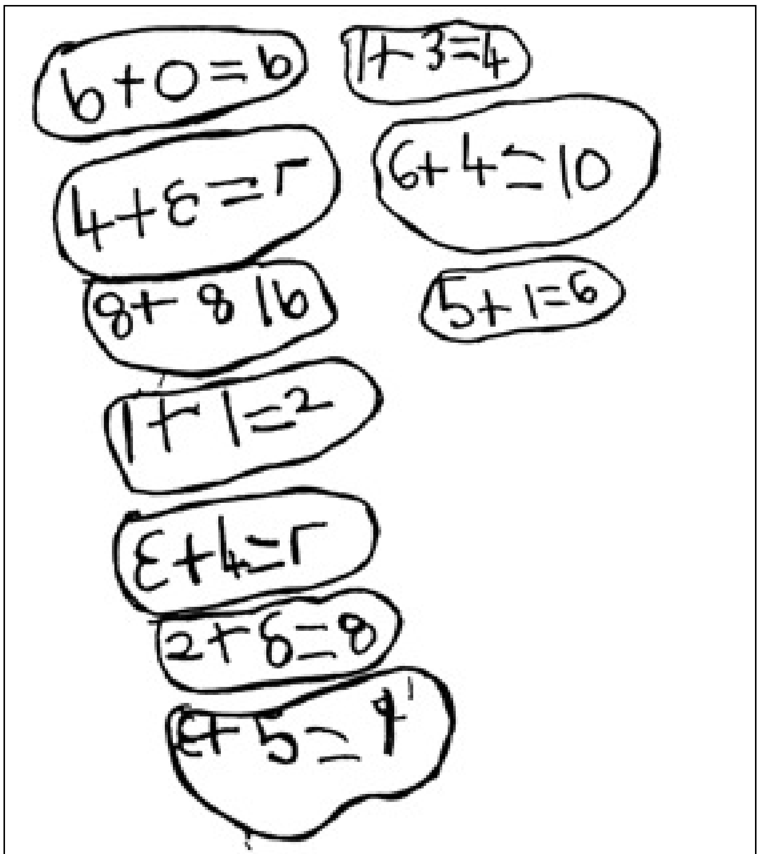

Figure 10.15 *Anna*

Calculating with larger numbers supported by jottings

Calculating with larger numbers is more challenging, since now children will need to have a feel for the larger numbers involved and may need to manipulate several steps. This is where mental methods and some taught 'jottings' can be of value.

For example, 7-year-old Miles's class were going on a residential trip and were to have a picnic on their journey. His teacher used this opportunity for some problem-solving, inviting the children to help work out how many trays (with three nectarines in each) they would need for 26 children. Miles had chosen to orientate his paper in 'portrait' format and quickly understood that he would run out of space on his empty number line: his highly adaptive solution was to double several of the jumps that he made as he worked from right to left.

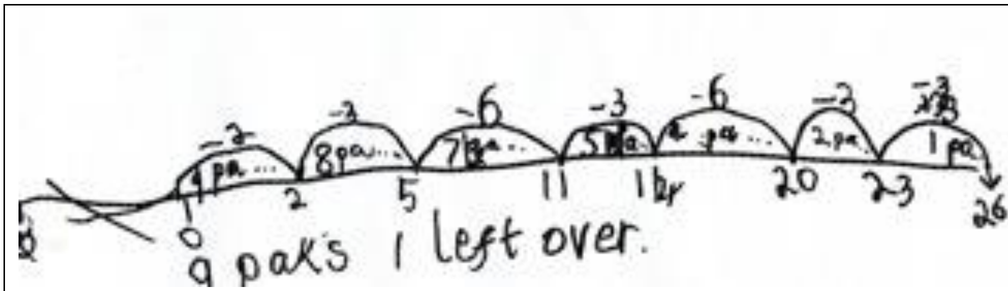

Figure 10.16 *Miles*

From this point in their development children move naturally to standard written mathematics and have developed deep levels of understanding calculations and problem-solving

Implications for teaching and learning

Despite recommendations from official government documents to support children's own mathematical thinking on paper, this is not happening. Whilst we have highlighted a number of implications for teaching and learning in this chapter, we would like to highlight three that we see as the most essential:

1. The foundations of early written mathematics begin in children's make-believe play and provide sure foundations for learning

2. Teachers need to consider that *all* young children's mathematical marks and representations are significant: their scribble-marks and early drawings for example, have as an important role in their development as later personal strategies children use in their calculations.

3. Finally, if children can calculate or solve a problem mentally they do not need to *record* the mathematics they have done. Recording places the emphasis on marks and drawings as a *product* and is a lower level of cognitive demand. The difference between recording a mathematical activity and *representing mathematical thinking* is one of quality and depth of thinking.

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